

DEFLECTION OF BEAMS

[Ref) Strength of material - DR.R.K.Bansal]

Deflection of A Beam Subjected To Uniform

Bending Moment:-

A beam AB of length 'L' is subjected to a uniform bending moment M as shown in fig. As the beam is subjected to a constant bending moment, hence it will bend into a circular arc. The initial position of the beam is

shown by ACB.

whereas the deflected position

is shown by AC'B.

let R=Radius of curvature of the deflected beam.

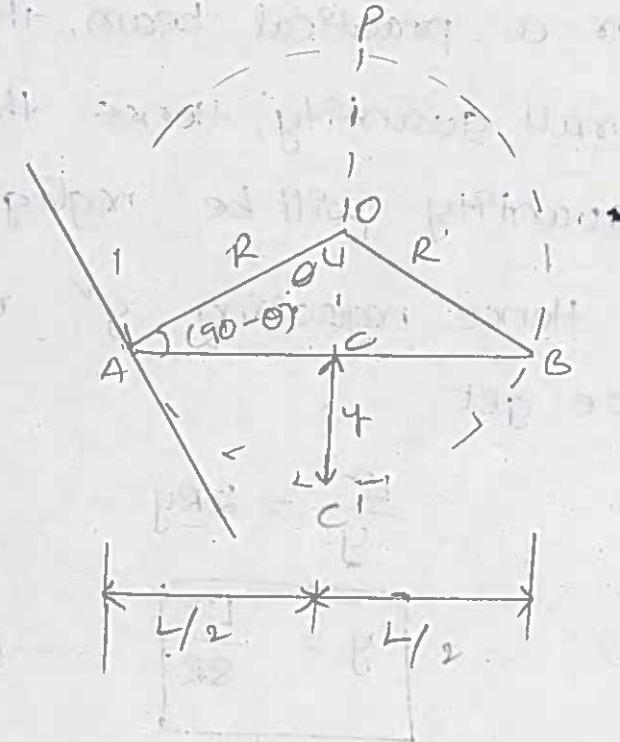
y = Deflection of the beam at the centre (i.e distance).

I = moment of inertia of the beam section

E = young's modulus for the beam

θ = slope of the beam at the end A.

for a practical beam the deflection 'y' is a small quantity.



Hence $\tan \Theta = \Theta$ where Θ is in radians. Hence

' Θ ' becomes the slope as slope is

$$\frac{dy}{dx} = \tan \Theta = \Theta$$

[In case of small deflection we can take $\tan \Theta \approx \Theta$]

$$\text{Now } AC = BC = \frac{L}{2}$$

Also from the geometry of a circle, we know that

$$AC \times CB = DC \times CC' \quad [DC = DC' - CC'] \\ \frac{L}{2} + \frac{L}{2} = (2R - y)xy \\ = 2R - y]$$

$$\frac{L^2}{4} = 2Ry - y^2$$

for a practical beam, the deflection 'y' is a small quantity, hence the square of a small quantity will be negligible.

Hence neglecting y^2 in the above equation we get

$$\frac{L^2}{4} = 2Ry$$

$$y = \frac{L^2}{8R} \quad \text{--- (1)}$$

But from bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

$$R = \frac{EI}{M} \quad \text{--- (2)}$$

(2) substituting the value of 'R' in equation,

① we get

$$y = \frac{L^2}{8 \times \frac{EI}{M}}$$

$y = \frac{ML^2}{8EI}$

The above equation gives the central deflection of beam which bends in a circular arc. value of slope (Θ)

from triangle AOB, we know that

$$\sin \Theta = \frac{AC}{AO} = \frac{(4_2)}{R} = \frac{L}{2R}$$

since the angle ' Θ ' is very small, hence $\sin \Theta = \Theta$ (in radians).

$$\therefore \Theta = \frac{L}{2R} \quad [\because R = \frac{EI}{M}]$$

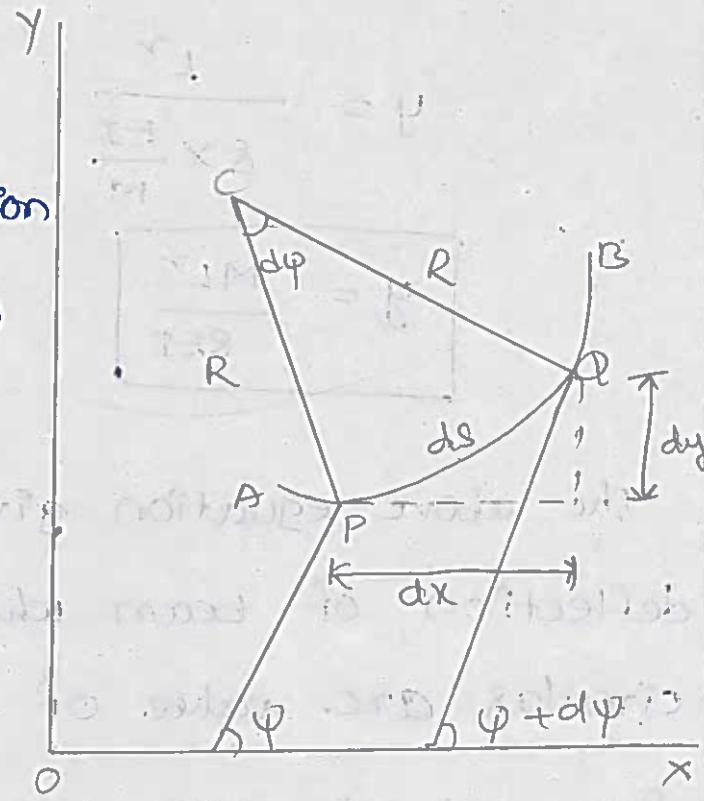
$$= \frac{L}{2 \times \frac{EI}{M}}$$

$\Theta = \frac{ML}{2EI}$

The above equation gives the slope of the deflected beam at 'A' or at 'B'.

Relation between slope, deflection and radius of curvature:

let the curve AB represents the deflection of a beam as shown in fig. Consider a small portion PQ of this beam. let the tangents at P and Q make angle ψ and $\psi + d\psi$ with x-axis.



Normal at 'P' and 'Q' will meet at 'C', such that

$$PC = QC = R$$

The point 'C' is known as centre of curvature of the curve PQ

let the length of PQ is equal to ds

from fig. we see that

$$\text{angle } PCQ = d\psi$$

$$PQ = ds = R \cdot d\psi$$

$$R = \frac{ds}{d\psi} \quad \text{--- (1)}$$

But if x and y be the co-ordinates of P, then

$$\tan \psi = \frac{dy}{dx} - \textcircled{2}$$

$$\sin \psi = \frac{dy}{ds}$$

$$\cos \psi = \frac{ds}{ds}$$

and Now equation $\textcircled{1}$ can be written as

$$R = \frac{ds}{d\psi} = \frac{(ds/dx)}{(d\psi/dx)}$$

$$R = \frac{\left(\frac{1}{\cos \psi}\right)}{\left(\frac{d\psi}{dx}\right)}$$

$$R = \boxed{\frac{\sec \psi}{\left(\frac{d\psi}{dx}\right)}} - \textcircled{3}$$

Differentiating equation $\textcircled{4}$ w.r.t. 'x' we get

$$\sec^2 \psi = \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\psi}{dx} = \frac{\left[\frac{d^2y}{dx^2}\right]}{\sec^2 \psi}$$

Substituting this value of $\frac{d\psi}{dx}$ in equation $\textcircled{3}$
we get

$$R = \frac{\sec \psi}{\left[\frac{\frac{d^2y}{dx^2}}{\sec^2 \psi}\right]} = \frac{\sec \psi \cdot \sec^2 \psi}{\left(\frac{d^2y}{dx^2}\right)}$$

$$R = \frac{\sec^3 \psi}{\left(\frac{d''y}{dx''} \right)}$$

Taking the reciprocal to both sides, we get

$$\frac{1}{R} = \frac{\frac{d''y}{dx''}}{\sec^3 \psi} = \frac{\frac{d''y}{dx''}}{(\sec^3 \psi)^{3/2}}$$

$$\frac{1}{R} = \frac{\frac{d''y}{dx''}}{(1 + \tan^2 \psi)^{3/2}}$$

for a practical beam, the slope $\tan \psi$ at any point is a small quantity. Hence $\tan^2 \psi$ can be neglected.

$$\boxed{\frac{1}{R} = \frac{d''y}{dx''}} - \textcircled{4}$$

from the bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\boxed{\frac{1}{R} = \frac{M}{EI}} - \textcircled{5}$$

equating $\textcircled{4} = \textcircled{5}$

$$\frac{M}{EI} = \frac{d''y}{dx''}$$

$$\boxed{M = EI \frac{d''y}{dx''}} - \text{Bending equation.}$$

differentiating the above equation w.r.t 'x'
we get

(4) $\frac{dM}{dx} = EI \cdot \frac{d^3y}{dx^3}$

$$\frac{dm}{dx} = F. (\text{shear force}).$$

$$F = EI \cdot \frac{d^3y}{dx^3} \rightarrow \text{shearing force}$$

differentiating above equation w.r.t x we get

$$\frac{df}{dx} = EI \frac{d^4y}{dx^4}$$

But $\frac{df}{dx} = \omega$

$$\therefore \omega = EI \frac{d^4y}{dx^4} \rightarrow \text{rate of loading.}$$

Methods for slope and deflection at a section:

Through there are many methods to find out the slope and deflection at a section in a loaded beam. Yet the following two methods are important.

1. Double integration method
2. Macaulay's method, and
3. Moment area method.

Double Integration method for slope and deflection:-

the bending moment at a point

$$M = EI \frac{d^2y}{dx^2}$$

integrating the above equation

$$\boxed{EI \cdot \frac{dy}{dx} = f_M} \quad \rightarrow ①$$

integrating the above equation once again

$$\boxed{EI y = \int \int M} \quad \rightarrow ②$$

it is thus obvious that after first integration the original differential equation, we get the value of slope at any point.

Simply Supported Beam with A central point Load

Load :-

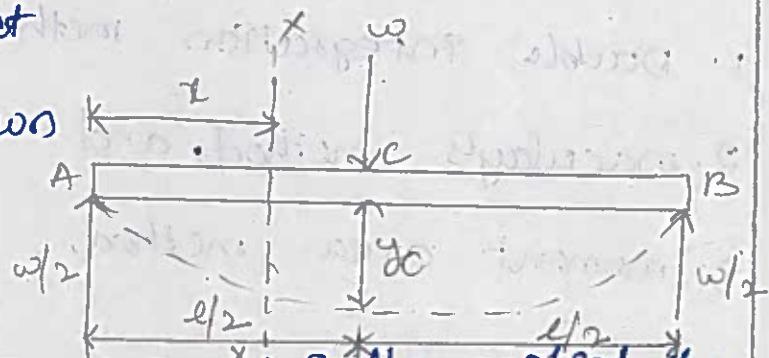
A simply supported beam AB of length 'L'

and carrying a

point load 'w' at

the centre is shown

in fig.



As the load is symmetrically applied the reactions R_A and R_B will be equal.

$$\text{Now } R_A = R_B = \frac{w}{2}$$

(5) Consider a section x at a distance x from A.

The bending moment at this section is given by

$$M_x = R_A \cdot x$$

$$M_x = \frac{\omega}{2} x^2$$

But B.M at any section is also given by equation as

$$M = EI \frac{dy}{dx^2}$$

equating the two value of B.M we get

$$EI \cdot \frac{dy}{dx^2} = \frac{\omega}{2} \frac{x^2}{2} + C_1 \quad \rightarrow ①$$

where C_1 is the constant of integration.

The boundary condition is that $x = \frac{L}{2}$;

$$\text{slop } \left(\frac{dy}{dx} \right) = 0$$

substitute the boundary values

$$0 = \frac{\omega}{4} \left[\frac{L}{2} \right]^2 + C_1$$

$$C_1 = -\frac{\omega L^2}{16}$$

substituting the value of C_1 in equation ①

$$EI \frac{dy}{dx} = \frac{\omega x^2}{4} - \frac{\omega L^2}{16} \rightarrow \text{slope} - ②$$

The above equation is known the slope equation. we find the slope at any point

on the beam, by substituting the values of x .

slope is maximum at A. at $A; x=0$;

hence slope at A will be obtained by substituting $x=0$ in the above equation.

$$EI \left(\frac{dy}{dx} \right)_{at A} = \frac{w}{4} (D)^2 - \frac{wL^2}{16}$$

$$EI \theta_A = -\frac{wL^2}{16}$$

$$\boxed{\theta_A = -\frac{wL^2}{16EI}}$$

the slope at point (B) will be equal to θ_A . since the load is symmetrically applied.

$$\boxed{\theta_B = \theta_A = -\frac{wL^2}{16EI}} \quad \text{- slope in radians.}$$

Deflection at any point:

Deflection at any point is obtained by integrating eqn ② the slope equation.

Hence

$$\boxed{EI \cdot y = \frac{w}{4} \frac{x^3}{3} - \frac{wL^2}{16} x + C_2}$$

at A, $x=0$; deflection is zero.

$$EI \cdot 0 = 0 - 0 + C_2$$

$$C_2 = 0$$

(6)

\therefore Substituting ' C_2 ' in above equation

$$EI \cdot y = \frac{\omega x^3}{12} - \frac{\omega L^3 x}{16}$$

the above equation is known as the deflection. the deflection is maximum at center point at C_1

where $x = \frac{L}{2}$; let y_c be the deflection at C_1

the substituting $x = \frac{L}{2}$; and $y = y_c$

we get

$$EI \cdot y_c = \frac{\omega}{12} \left[\frac{L}{2} \right]^3 - \frac{\omega L^3}{16} \left[\frac{L}{2} \right]$$

$$= \frac{\omega L^3}{96} - \frac{\omega L^3}{32}$$

$$= \frac{\omega L^3 - 3\omega L^3}{96}$$

$$y_c = -\frac{\omega L^3}{48 EI}$$

(-ve sign shows that the deflection is down wards)

\therefore Down ward deflection

$$y_c = \frac{\omega L^3}{48 EI}$$

Deflection of A simply supported Beam

with A UDL:-

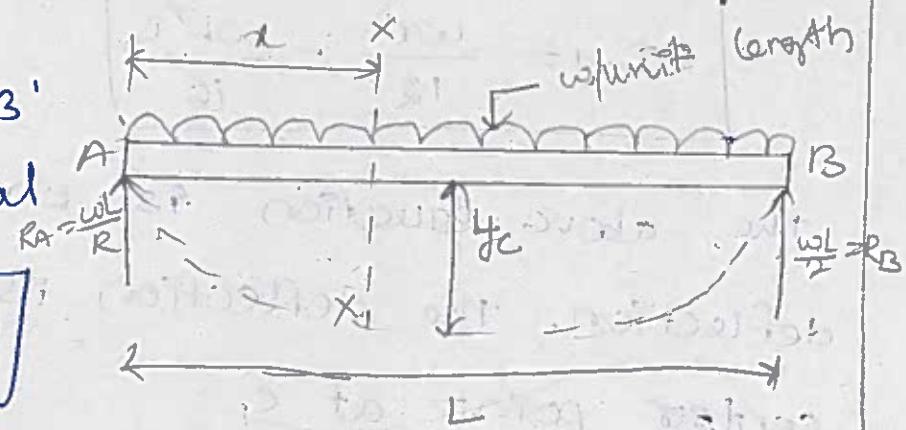
simply supported beam of length 'L'

and carrying a UDL of u/unit length

over the entire length is shown in fig. The reactions at 'A' and 'B' will be equal

$R_A = \frac{wL}{2}$

$$R_B = \frac{wL}{2}$$



Consider a section 'x' at a distance 'x' from A. The bending moment at this section is given by

$$M_x = R_A x - w \cdot x \cdot \frac{x}{2}$$

But B.M at any section is also given by equation.

$$M = EI \frac{dy}{dx^2}$$

equating the two values of B.M, we get

$$EI \frac{dy}{dx} = \frac{wL}{2} \cdot \frac{x^2}{2} - \frac{w x^3}{2 \cdot 3} + C_1$$

where C_1 is a constant of integration

Integrating the above equation again, we get.

$$EI y = \frac{wL}{4} \cdot \frac{x^3}{3} - \frac{w}{6} \frac{x^4}{4} + C_1 x + C_2 \quad \text{---(2)}$$

where C_2 is a constant of integration.

thus two constants of integration (i.e. c_1 and c_2) are obtained from boundary conditions.

i) at $x=0$; $y=0$ and ii) at $x=L$; $y=0$

substituting first boundary condition i.e $x=0$, $y=0$ in equation ②

$$0 = 0 - 0 + 0 + c_2$$

$$\boxed{c_2 = 0}$$

substituting the second boundary condition i.e at $x=L$; $y=0$ in equation ② we get

$$0 = \frac{\omega \cdot L}{4} \cdot \frac{L^3}{3} - \frac{\omega}{6} \cdot \frac{L^4}{4} + c_1 L + c_2$$

$$0 = \frac{\omega L^4}{12} - \frac{\omega L^4}{24} + c_1 L + 0 \quad [\because c_2 = 0]$$

$$c_1 = -\frac{\omega L^3}{12} + \frac{\omega L^3}{24}$$

$$\boxed{c_1 = -\frac{\omega L^3}{24}}$$

substitute the value of c_1 in equation ① & ② we get

$$\boxed{EI \cdot \frac{dy}{dx} = \frac{\omega \cdot L}{4} x^3 - \frac{\omega x^3}{6} - \frac{\omega L^3}{24}} \quad \text{slope}$$

$$\text{and } EI \cdot y = \frac{\omega L}{12} x^2 - \frac{\omega}{24} x^4 + \left[-\frac{\omega L^3}{24} \right] x + 0$$

$$EIy = \frac{\omega L x^3}{12} - \frac{\omega}{24} x^4 - \frac{\omega L^3 x}{24} \rightarrow \text{Deflection}$$

at A; $x=0$; and $\frac{dy}{dx} = \Theta_A$

substitute the value in slope equation

$$EI\left(\frac{dy}{dx}\right)_{\text{at } A} = \frac{\omega \cdot L}{4}(0)^3 - \frac{\omega(0)^3}{6} - \frac{\omega L^3}{24}$$

$$EI \Theta_A = -\frac{\omega L^3}{24} \quad [\because \omega L = w]$$

$$EI \Theta_A = -\frac{\omega L^3}{24} \quad [\because \omega L = w]$$

$$EI \Theta_A = -\frac{\omega L^3}{24}$$

$$\Theta_A = -\frac{\omega L^3}{24EI}$$

$$\text{By symmetry } \Theta_B = -\frac{\omega L^3}{24EI}$$

The maximum deflection is at the centre of the beam i.e. at point C, where

$$x = \frac{L}{2};$$

let y_C = deflection at C.

$$\therefore EIy_C = \frac{\omega \cdot L}{12} \left[\frac{L}{2}\right]^3 - \frac{\omega}{24} \left[\frac{L}{2}\right]^4 - \frac{\omega L^3}{24} \left[\frac{L}{2}\right]$$

$$= \frac{\omega L^3}{96} - \frac{\omega L^4}{384} - \frac{\omega L^4}{48}$$

$$= -\frac{5\omega L^4}{384}$$

②

$$y_c = -\frac{5wL^4}{384}$$

$\therefore w \cdot L = w = \text{total load}$

\therefore Downward deflection

[+ve sign indicate

deflection is downwards]

$$y_c = \frac{5}{384} \frac{wL^3}{EI}$$

Simply Supported Beam with A Gradually Varying Load:-

Consider a simply

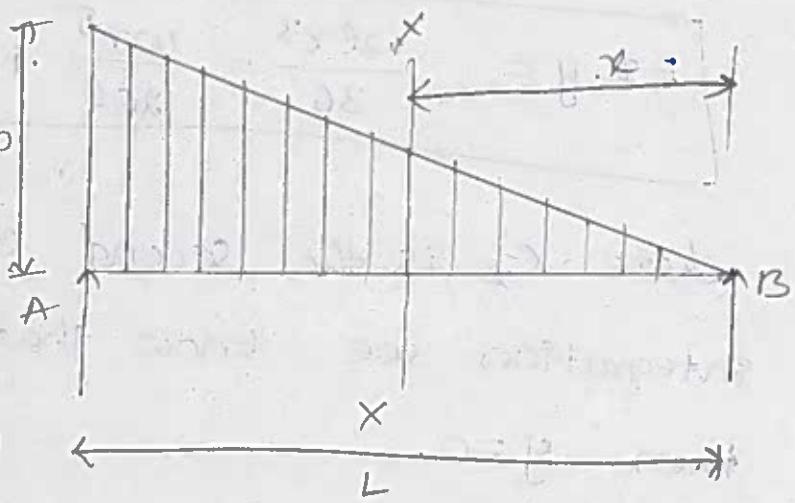
supported beam AB

of length 'L'

carrying a gradually varying

load from zero at

B' to $w/\text{unit length}$ at A as shown in fig.



$$\therefore R_A = \frac{wl}{3} \text{ and } R_B = \frac{wl}{6}$$

Now consider a section x at a distance x from B. we know that the bending moment at this section.

$$M_x = R_B \cdot x - \left[\frac{wx}{l} \times \frac{x}{2} \times \frac{x}{3} \right]$$

$$M_x = \frac{wx}{6} - \frac{wx^3}{6l}$$

$$\therefore EI \frac{d^2y}{dx^2} = \frac{\omega l x}{6} - \frac{\omega x^3}{6l} \quad \text{--- (1)}$$

integrating the above equation.

$$EI \frac{dy}{dx} = \frac{\omega l x^2}{12} - \frac{\omega x^4}{24l} + C_1 \quad \text{--- (2)}$$

where C_1 is the first constant of integration, integrating the equation (2) once again.

$$EI \cdot y = \frac{\omega l x^3}{36} - \frac{\omega x^5}{120l} + C_1 x + C_2 \quad \text{--- (3)}$$

where C_2 is the second constant of integration we know that when $x=0$, then $y=0$.

$$\therefore C_2 = 0$$

we also know that when $x=l$; then $y=0$; substituting these values in equation (3)

$$\therefore 0 = \frac{\omega l}{36} l^3 - \frac{\omega}{120l} \times l^5 + C_1 l$$

$$0 = \frac{\omega l^4}{36} - \frac{\omega l^4}{120} + C_1 l$$

$$\therefore C_1 = \frac{\omega l^3}{36} - \frac{\omega l^3}{120}$$

$$C_1 = -\frac{7\omega l^3}{360}$$

①

Now substituting this value of c_1 in equation ②

$$EI \frac{dy}{dx} = \frac{\omega l x^2}{12} - \frac{\omega x^4}{24l} - \frac{7\omega l^3}{360} \quad \text{--- (4)}$$

thus for slope at A substituting $x=l$ in above equation (4)

$$EI \theta_A = \frac{\omega l l^2}{12} - \frac{\omega l^4}{24l} - \frac{7\omega l^3}{360}$$

$$EI \cdot \theta_A = \frac{\omega l^3}{12} - \frac{\omega l^3}{24} - \frac{7\omega l^3}{360}$$

$$\theta_A = - \frac{\omega l^3}{48EI}$$

Now for slope at B; substitute $x=0$ in equation (4)

$$E.I. \theta_B = - \frac{7\omega l^3}{360}$$

$$\theta_B = - \frac{7\omega l^3}{360EI}$$

Now substituting the value of c_1 in equation (3)

$$\therefore EI \cdot y = \frac{\omega l x^3}{36} - \frac{\omega x^5}{120l} - \frac{7\omega l^3 x}{360}$$

$$y = \frac{1}{EI} \left[\frac{\omega l x^3}{36} - \frac{\omega x^5}{120l} - \frac{7\omega l^3 x}{360} \right]$$

for deflection at the centre of the beam

substituting $x = \frac{l}{2}$

$$y_c = \frac{1}{EI} \left[\frac{\omega l}{36} \left(\frac{l}{2}\right)^3 - \frac{\omega}{120l} \left(\frac{l}{2}\right)^5 - \frac{7\omega l^3}{360} \left(\frac{l}{2}\right) \right]$$

$$y_c = - \frac{0.00651 \omega l^4}{EI}$$

[\therefore -ve sign due to
downward
deflection]

$$\therefore y_c = \frac{0.00651 \omega l^4}{EI}$$

we know that the maximum deflection will occur, where slope of the beam is zero. Therefore equating the equation (4) to zero.

$$\frac{\omega l x^2}{12} - \frac{\omega x^4}{24l} - \frac{7\omega l^3}{360} = 0$$

$$x = 0.319l$$

Now substituting this value of x in equation.

$$\therefore y_{max} = \frac{1}{EI} \left[\frac{\omega l}{36} (0.319l)^3 - \frac{\omega}{120l} (0.319l)^5 - \frac{7\omega l^3}{360} (0.319l) \right]$$

$$y_{max} = - \frac{0.00652 \omega l^4}{EI} \quad (\because -ve sign due
to downward)$$

$$\therefore y_{max} = \frac{0.00652 \omega l^4}{EI}$$

(10) Double Integration Method:-

Cantilevers:-

I. Cantilever beam with concentrated at free end:-

Fig shows a cantilever with concentrated load.

w acting at free end.

Let the moment of inertia of the section of the

cantilever about the neutral axis, be I . Consider a section xx' at a distance ' x ' from the fixed end A.

$$\therefore Mx = -w(l-x)$$

$$\therefore EI \frac{dy}{dx} = -w(l-x)$$

Integrating we get

$$EI \frac{dy}{dx} = -w\left(lx - \frac{x^2}{2}\right) + C_1$$

where C_1 = constant of integration

$$\therefore \text{at } x=0; \frac{dy}{dx} = 0$$

$$\therefore C_1 = 0$$

$$\therefore EI \frac{dy}{dx} = -w \left(lx - \frac{x^2}{2}\right) \rightarrow ①$$

Slope at B; putting $x=l$, we have

$$\Theta_B = \frac{dy}{dx} = -\frac{1}{EI} w \left[l \cdot l - \frac{l^3}{2} \right]$$
$$= \frac{\omega l^2}{2EI}$$

i.e $\boxed{\Theta_B = -\frac{\omega l^2}{2EI}}$

To get deflection, integrating eqn① above, we get

$$EI \cdot y = -\omega \left[l \frac{x^2}{2} - \frac{x^3}{6} \right] + C_2$$

at A (fixed end)

$$\text{at } x=0; y=0; C_2=0$$

$$EI \cdot y = -\omega \left[l \frac{x^2}{2} - \frac{x^3}{6} \right] \quad \text{② deflection equation}$$

Deflection at B; putting $x=l$; we get

$$y_B = -\frac{\omega}{EI} \left[l \frac{l^2}{2} - \frac{l^3}{6} \right]$$

$$= -\frac{\omega l^3}{3EI}$$

\therefore Down ward deflection of 'B'

$\boxed{y_B = \frac{\omega l^3}{3EI}}$ C-ve sign indicates
deflection is down
wards)

(ii) Cantilever of Length 'l' carrying UDL of w per unit Run over whole length :-

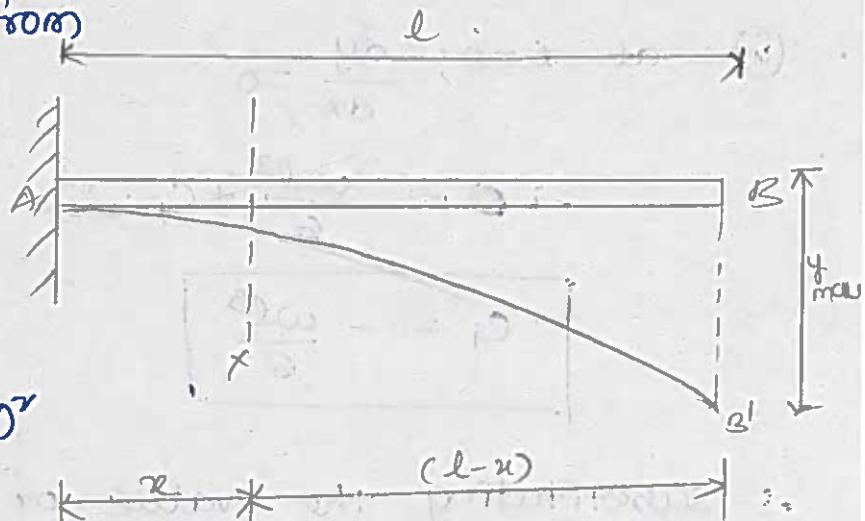
Consider a section 'x x'

at a distance x from

fixed end A.

$$M_x = -\frac{w(l-x)^2}{2}$$

$$EI \frac{d^2y}{dx^2} = -\frac{w(l-x)^2}{2}$$



Integrating, we get

$$EI \cdot \frac{dy}{dx} = -\frac{w}{2} \left(\frac{(l-x)^3}{3} (-1) + C_1 \right)$$

$$EI \frac{dy}{dx} = +\frac{w}{6} (l-x)^3 + C_1$$

$$\text{at fixed end A; } x=0; \frac{dy}{dx} = 0$$

$$C_1 = -\frac{wl^3}{6}$$

$$EI \frac{dy}{dx} = +\frac{w}{6} (l-x)^3 - \frac{wl^3}{6} \quad \text{--- (1) slope equation}$$

Slope at 'B' putting $x=l$; we have

$$EI \cdot \Theta_B = \frac{dy}{dx} = \frac{w}{6} (l-l)^3 - \frac{wl^3}{6}$$

$$\therefore \Theta_B = -\frac{wl^3}{6EI}$$

To get deflection integrating (1) we get

$$EI \cdot y = +\frac{w}{6} \frac{(l-x)^4}{4} (-1) + C_1 x + C_2$$

$$\therefore 0 = -\frac{\omega l^4}{24} + C_2$$

$$C_2 = \frac{\omega l^4}{24}$$

(ii) at $x=0$; $\frac{dy}{dx} = 0$

$$\therefore 0 = \frac{\omega l^3}{6} + C_1$$

$$C_1 = -\frac{\omega l^3}{6}$$

∴ substituting the value of C_1 and C_2 in α
above equal

$$EI \cdot y = -\frac{\omega(l-x)^4}{24} - \frac{\omega l^3}{6}x + \frac{\omega l^4}{24} - \textcircled{2}$$

∴ Deflection at B; deflect equal

putting $x=l$ we get

$$EI \cdot y_B = -\frac{\omega}{24}(l-l)^4 - \frac{\omega l^3}{6} \cdot l + \frac{\omega l^4}{24}$$

$$= -\frac{\omega l^4}{6} + \frac{\omega l^4}{24} = -\frac{\omega l^4}{8}$$

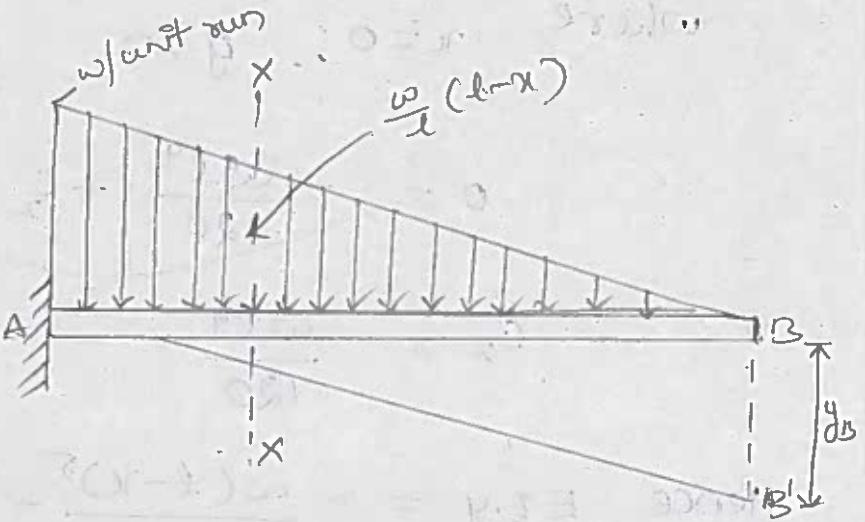
$$\therefore y_B = -\frac{\omega l^4}{8 EI} \rightarrow \text{downward deflection at } B.$$

-ve sign indicates
deflection is down wards.

(17)

Cantilever of Length 'l' carrying A GVL
From zero At the free End To w per
Unit Run At the Fixed End &

from fig consider
a section xx at a
distance ' x ' from
the fixed end



Intensity of
loading at the
section xx
 $= \frac{w}{l}(l-x)$ per unit
run.

The B.M of the section xx .

$$M_x = -\frac{1}{2} (l-x) \times \frac{w}{l} (l-x) \cdot \frac{(l-x)}{3}$$

$$\therefore EI \frac{d^2y}{dx^2} = -\frac{w(l-x)^3}{6l}$$

Integrating for slope, we get

$$EI \frac{dy}{dx} = + \frac{w(l-x)^4}{24l} + C_1 \quad [\text{where } C_1 = \text{constant}]$$

$$\text{at } x=0; \frac{dy}{dx} = 0; C_1 = -\frac{wl^3}{24}$$

$$\text{Here } EI \frac{dy}{dx} = + \frac{w(l-x)^4}{24l} - \frac{wl^3}{24} \quad \text{① Slope equation}$$

Slope at 'B' putting $x=l$; we get

$$OB = \frac{dy}{dx} = -\frac{wl^3}{24EI}$$

To get deflection, integrating the equation ① we get

$$EIy = -\frac{\omega(l-x)^5}{120l} - \frac{\omega l^3}{24}x + C_2 \quad [C_2 = \text{constant}]$$

where $x=0; y=0$

$$0 = -\frac{\omega l^4}{120} + C_2$$

$$C_2 = \frac{\omega l^4}{120}$$

Hence $EI.y = -\frac{\omega(l-x)^5}{120l} - \frac{\omega l^3}{24}x + \frac{\omega l^4}{120}$ ②

Deflection at 'B' putting $x=l$; equation

we get

$$EI.y_B = -\frac{\omega l^4}{24} + \frac{\omega l^4}{120}$$

$$= -\frac{\omega l^4}{80}$$

$$y_B = -\frac{\omega l^4}{30EI}$$

↓ is downward deflection at 'B'

iv) case:- Cantilever of length 'l' carrying

A G.V.R whose intensity varies zero

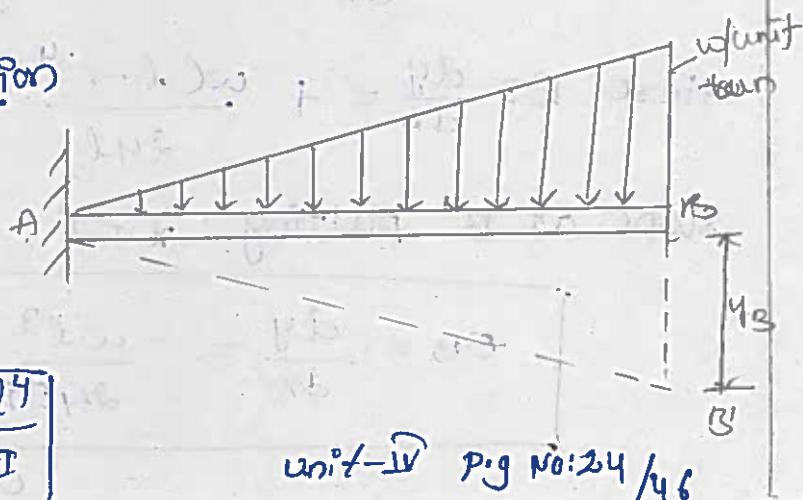
At fixed end w/unit Run At Free End:-

Down ward deflection

at 'B'

$$y_B = \frac{\omega l^4}{8EI} - \frac{\omega l^4}{30EI}$$

$$\therefore y_B = \frac{11\omega l^4}{120EI}$$



Macaulay's Method:

The procedure of finding beam with an eccentric point load.

This method was derived by Mr. H Macaulay and is known as macaulay's method.

A simply supported

beam 'B' of length 'L'

and carrying a point

load 'w' at a distance

'a' from left support

and at a distance 'b'

from right support is shown in fig.

The reactions at 'A' and 'B' are given by

$$R_A = \frac{w \cdot b}{L} \text{ and } R_B = \frac{w a}{L}$$

The bending moment at any sections b/w 'A' and 'C' at a distance 'x' from 'A' is given by

$$M_x = R_A x = \frac{w b}{L} x$$

The above equation of B.M holds good for the values of 'x' between '0' and 'a'. The B.M at any section between 'C' and 'B' at distance 'x' from 'A' given

$$M_x = R_A x - w(x-a)$$

$$M_x = \frac{wb}{L}x - w(x-a)$$

the above equation of B.M holds good for all values of 'x' between $x=a$ and $x=b$.

the B.M for all sections of the beam can be expressed in a single equation written as

$$M_x = \frac{wb}{L}x \cdot | -w(x-a) | - \textcircled{1}$$

stop at the dotted line for any point in section AC. But for any point in section CB, add the expression beyond the dotted line also

the B.M at any section is also given by

$$M = EI \frac{d^2y}{dx^2} - \textcircled{2}$$

from $\textcircled{1}$ & $\textcircled{2}$ we get

$$EI \frac{d^2y}{dx^2} = \frac{wb}{L}x \cdot | -w(x-a) | - \textcircled{3}$$

Integrating the above equation we get

$$EI \frac{dy}{dx} = \frac{wb}{L} \frac{x^2}{2} + C_1 - w \frac{(x-a)^2}{2} - \textcircled{3}$$

where ' C_1 ' is a constant of integration.

this constant of integration should be written after the first term also the brackets are to be integrated as a whole.

Hence the integration of $(x-a)$ will be $\frac{(x-a)^2}{2}$
and not $x^2 - ax$

integrating equation (4) were again we get

$$EI.y = \frac{wb}{2L} \cdot \frac{x^3}{3} + C_1 x + C_2 - \frac{\omega(x-a)^3}{2 \cdot 3} \quad (5)$$

where C_2 is another constant of integration
this constant is written after $C_1 x$. The
integration of $(x-a)^2$ will be $\left[\frac{x-a}{3}\right]^3$. this
type of integration is justified at the
constant of integrations C_1 and C_2 are
valid for all values of x .

The values of C_1 and C_2 are obtained
from boundary conditions. two boundary
conditions are

i) at $x=0; y=0$ and ii) At $x=L; y=0$
up to dotted line only we get

$$0 = 0 + 0 + C_2$$

$$C_2 = 0$$

ii) at B; $x=L; y=0$: substitute these value
in equation we get

$$0 = \frac{wb}{2L} \cdot \frac{L^3}{3} + C_1 x L + 0 - \frac{\omega}{2} \cdot \frac{(L-a)^3}{3}$$

$$0 = \frac{wbL^2}{6} + C_1 L - \frac{wb^3}{2 \cdot 3} \quad [\because C_2 = 0] \quad [\because L-a=b]$$

$$C_1 L = \frac{wb^3}{6} - \frac{wbL^2}{6}$$

$$C_1 L = -\frac{wb}{6} (L^2 - b^2)$$

$$c_1 = -\frac{wb}{GL} (L^2 - b^2) \quad \textcircled{6}$$

substituting the value of c_1 in equation

(4) we get

$$EI \cdot \frac{dy}{dx} = \frac{wb}{L} \cdot \frac{x^2}{2} + \left[-\frac{wb}{GL} (L^2 - b^2) \right] + \frac{w(x-a)^2}{2}$$

$$EI \frac{dy}{dx} = \frac{wbx^2}{2L} - \frac{wb}{GL} (L^2 - b^2) - \frac{w(x-a)^2}{2} \quad \textcircled{7}$$

equation (7) gives the slope at any point in the beam, slope is maximum at A or B. To find the slope at A. Substitute $x=0$; in the above equation upto dotted line as point A lies in AC.

$$\begin{aligned} \therefore EI \cdot \Theta_A &= \frac{wb}{2L} \times 0 - \frac{wb}{GL} (L^2 - b^2) \\ &= -\frac{wb}{GL} (L^2 - b^2) \end{aligned}$$

[∴ $\frac{dy}{dx}$ at
 $A = \Theta_A$].

$$\Theta_A = -\frac{wb}{6EI} (L^2 - b^2)$$

substituting the values of c_1 and c_2 in equation (5) we get

$$EI \cdot y = \frac{wb}{GL} x^3 + \left[-\frac{wb}{6L} (L^2 - b^2) \right] + 0 + \frac{w(x-a)^3}{6} \quad \textcircled{8}$$

equation (8) gives the deflection at any point in the beam to find the deflection at end under the load, substitute $x=a$ in equation (8) and consider the equation

(15)

upto deflected line (as per given conditions)

Here, we get

$$EI \cdot y_c = \frac{wb}{6L} \cdot a^3 - \frac{wb}{6L} (L^3 - b^3) a$$

$$= \frac{wb}{6L} a [a^2 - L^2 + b^2]$$

$$[\because L = a + b]$$

$$= - \frac{wab}{6L} (L^2 - a^2 - b^2)$$

$$= - \frac{wab}{6L} [(a+b)^2 - a^2 - b^2]$$

$$= - \frac{wab}{6L} [a^2 + b^2 + 2ab - a^2 - b^2]$$

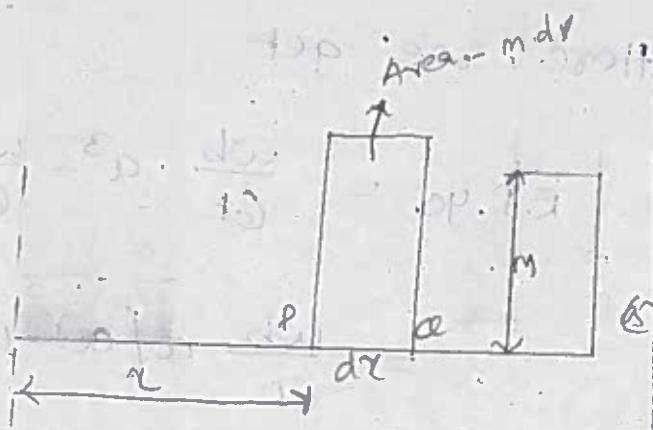
$$= - \frac{wab}{6L} [2ab]$$

$$EI \cdot y_c = - \frac{wa^2 b^2}{3L}$$

$$\therefore y_c = - \frac{wa^2 b^2}{SEIL}$$

The moment area method is partially convenient if areas of beams acted upon with point loads in which case bending moment area consists of triangular and rectangular.

fig shows a beam AB carrying some type of loading, and hence subjected to bending moment as shown in fig (a).



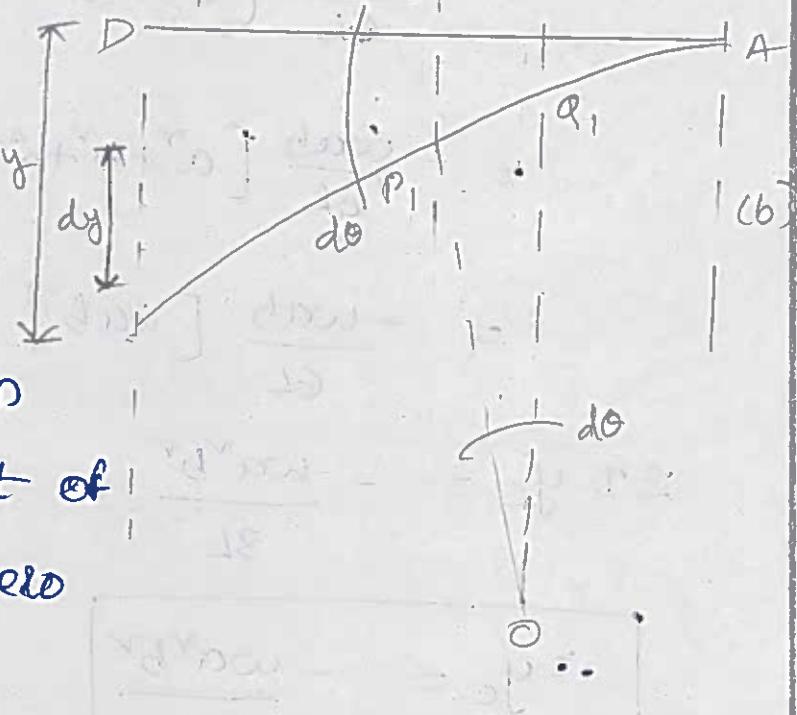
Let the beam bent

into

AQP,B as shown in fig.

Due to the load acting on the beam

let A be a point of zero slope and zero deflection.



consider an element PQ of small length dx at a distance x from B.

The corresponding points on the deflected beam are P_1Q_1 as shown in fig (b)

let R = Radius of curvature of deflected point P_1Q_1 .

$d\theta$ = angle subtended by the arc P_1Q_1 at the centre O

(6)

M = Bending moment between P and Q

AC = tangent at point P_1 ,

$Q_1 D$ = tangent at point Q_1 ,

the tangent at P_1 and Q_1 , cutting the vertical line through 'B' at points 'C' and D. The angle between the normals at P_1 and Q_1 will be equal to the angle between tangents $P_1 C$ and $Q_1 D$. Hence the angle between the lines CQ_1 and $P_1 Q_1$ will be equal to θ .

for the deflected part $P_1 Q_1$ of the beam, we have

$$P_1 Q_1 = R \cdot d\theta$$

$$\text{But } P_1 Q_1 = dx$$

$$dx = R \cdot d\theta$$

$$d\theta = \frac{dx}{R} \quad \dots \textcircled{1}$$

But for a loaded beam, we have

$$\frac{M}{I} = \frac{E}{R} \quad (\text{or}) \quad R = \frac{EI}{M}$$

Substituting the values of ' R ' in equation

① we get.

$$d\theta = \frac{dx}{\left(\frac{EI}{M}\right)}$$

$$d\theta = \frac{M dx}{EI} - \textcircled{D}$$

Since the slope at point A is assumed zero, hence total slope at B is obtained by integration the above equation between the limits '0' and 'L'.

$$\theta = \int_0^L \frac{M dx}{EI}$$

$$\theta = \frac{1}{EI} \int_0^L M dx$$

But $M \cdot dx$ represents the area of B.M diagram of length dx .

Hence $\int_0^L M dx$ represents the area of B.M diagram between 'A' and 'B'.

$\therefore \theta = \frac{1}{EI} [\text{Area of B.M diagram between A and B}]$

But $\theta = \text{slope at B} = \theta_B$

$\therefore \text{slope at B.}$

$$\theta_B = \frac{\text{Area of B.M diagram between A & B}}{EI}$$

If the slope at 'A' is not zero then,

we have

"Total change of slope between B and A

is equal to the area of B.M diagram between B and A divided by the flexural rigidity EI"

$$\Theta_B - \Theta_A = \frac{\text{Area of B.M between A & B}}{EI}$$

Now the deflection, due to bending of the portion P, Q, RS

$$dy = x \cdot d\theta$$

Substituting the value of $d\theta$ from equation

② we get

$$dy = x \frac{M \cdot dx}{EI} \quad \text{--- } ③$$

Since deflection at A is assumed to be zero, hence the total deflection at B is obtained by integrating the above equation between the limits zero and 'L'

$$y = \int_0^L x \frac{M dx}{EI} = \frac{1}{EI} \int_0^L x M dx$$

But $x \cdot M dx$ represents the moment of area of the B.M diagram of length dx about point B.

Hence $\int x M dx$ represents the moment of area of the B.M diagram between B and A about B.

This is equal to the total area of B.M diagram between B and A multiplied by the distance of the C.G of the B.M diagram area from B.

$$\therefore y = \frac{1}{EI} A \cdot \bar{x}$$

$$y = \frac{A\bar{x}}{EI}$$

where A = Area of the B.M diagram between A and B

\bar{x} = Distance of C.G of the area A from B.

Mohr's theorem

1. The change of slope between any two points is equal to the net area of the B.M diagram between these points divided by EI.
2. The total deflection between any two points is equal to the moment of the area of B.M diagram between the two points about the last point (rcB)

(B) divided by EI.

The Mohr's theorems is conveniently used in following cases.

1. problems on cantilevers (zero slope at fixed end)
2. simply supported beams carrying symmetrical loading (zero slope at the centre).
3. Beams fixed at both ends (zero slope at each end).

The B.M diagram is a parabola for uniformly distributed loads. The following properties of area and centroids of parabola are given as

$$\text{let } BC = d$$

$$AB = b$$

from fig ABC is a parabola and ABCD is a surrounding rectangle.

$$\text{let } A_1 = \text{Area of } ABC$$

$$\bar{x}_1 = \text{Distance of C.G } A_1 \text{ from AD}$$

$$A_2 = \text{Area of ACD}$$

$$\bar{x}_2 = \text{Distance of C.G of } A_2 \text{ from AD}$$

$$G_1 = \text{C.G of Area } A_1$$

$$G_2 = \text{C.G of Area } A_2$$

then $A_1 = \text{Area of parabola } ABC = \frac{2}{3} bd^2$

$A_2 = \text{Area } ACD = \text{Area } ABCD - \text{Area } ABC$

$$= b \times d - \frac{2}{3} bd^2$$

$$= \frac{1}{3} bd$$

$$\bar{x}_1 = \frac{5}{8} b$$

$$\bar{x}_2 = \frac{1}{4} b.$$

(Case(i)):- Cantilever Beam with a concentrated Load

At free End:-

* cantilever with a concentrated load

'w' acting at free end the elastic curve and B.M diagram respectively.

the slope and deflection will be maximum at the free end, we know that

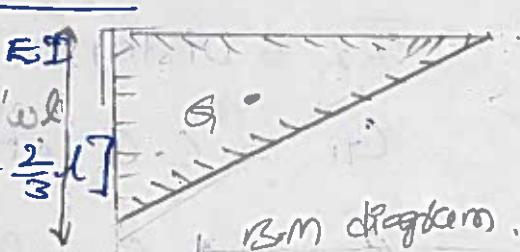
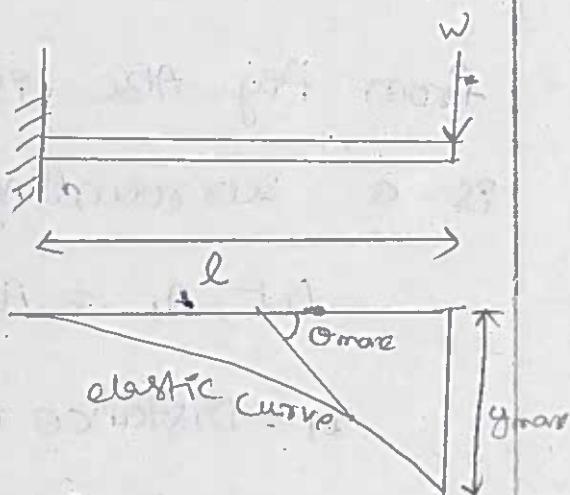
$$\Theta_{\max} = \frac{A}{EI}$$

$$A = \frac{1}{2} \times l \times w l = \frac{wl^2}{2}$$

$$\Theta_{\max} = \frac{wl^3}{2EI}$$

$$\text{also } y_{\max} = \frac{Ax}{EI} = \frac{wl^3 \times \frac{2l}{3}}{EI} = \frac{2wl^6}{9EI}$$

$$= \frac{wl^3}{3EI} \quad [\because x = \frac{2}{3}l]$$



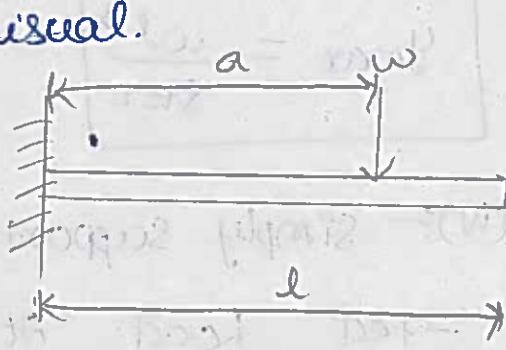
19

then

$$y_{\max} = \frac{w l^3}{3EI}$$

(case(ii)) - Cantilever Beam with A concentrated load At Any point:

The slope and deflection will be maximum at the free end as usual.



$$\theta_{\max} = \frac{A}{EI}$$

$$\text{But, } A = \frac{1}{2} a \cdot w a = \frac{w a^2}{2}$$

$$\theta_{\max} = \frac{w a^3}{2EI}$$

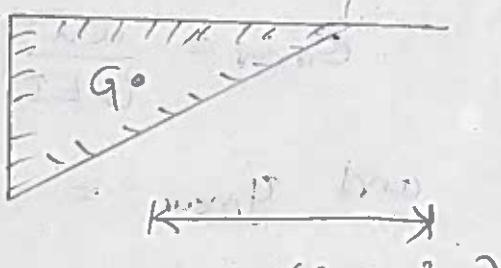
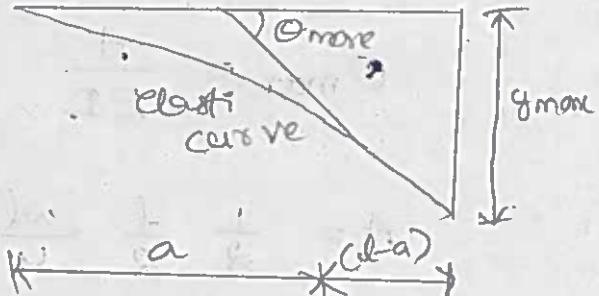
$$\text{and } y_{\max} = \frac{A x}{EI} = \frac{w a^2}{2EI}$$

$$\frac{A x}{EI} = \frac{w a^2}{2EI} \left[l - a + \frac{2}{3}a \right]$$

$$y_{\max} = \frac{w a^2}{2EI} \left[l - \frac{a}{3} \right]$$

$$y_a = \frac{A x}{EI} = \frac{w a^2}{2EI} \left(\frac{2}{3}a \right)$$

$$y_a = \frac{w a^3}{3EI}$$



B.M diagram

(case(iii)) - Cantilever Beam with Uniformly Distributed Load:-

Total load on the

$$\text{cantilever} = w l = w$$

$$\text{Now } \theta_{\max} = \frac{A}{EI}$$

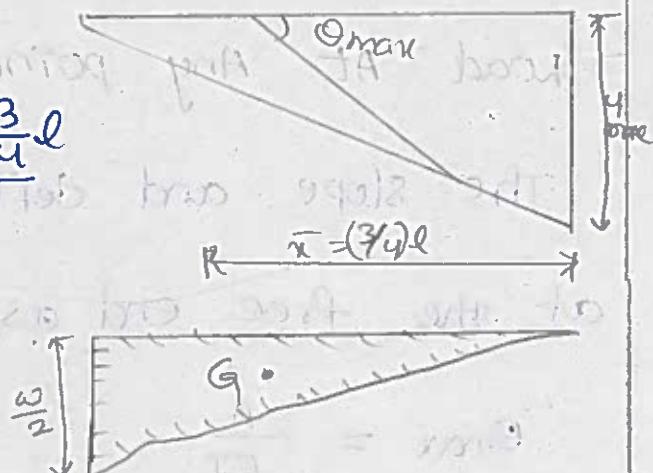
$$\text{But } A = \frac{1}{3} l \cdot \frac{\omega l}{2} = \frac{\omega l^3}{6}$$

$$\Theta_{\max} = \frac{\omega l^2}{G E I}$$

and

$$y_{\max} = \frac{A \bar{x}}{E I} = \frac{\omega l^3 \times \frac{3}{4} l}{E I}$$

$$y_{\max} = \frac{\omega l^3}{8 E I}$$



case (iv): simply supported Beam with Concentra
-ted load At the

$$\Theta_{\max} = \frac{A}{E I}$$

$$A = \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{\omega l}{4} = \frac{\omega l^3}{16}$$

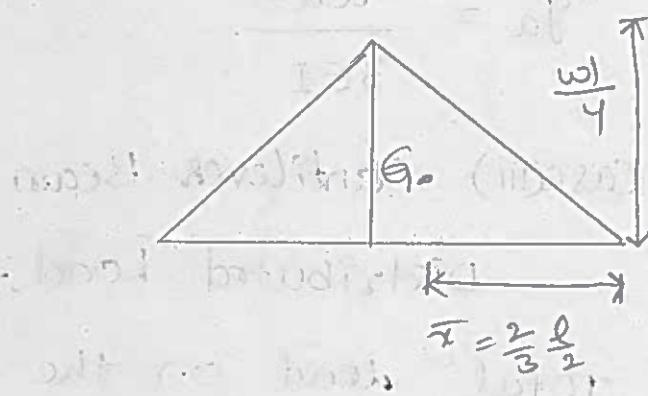
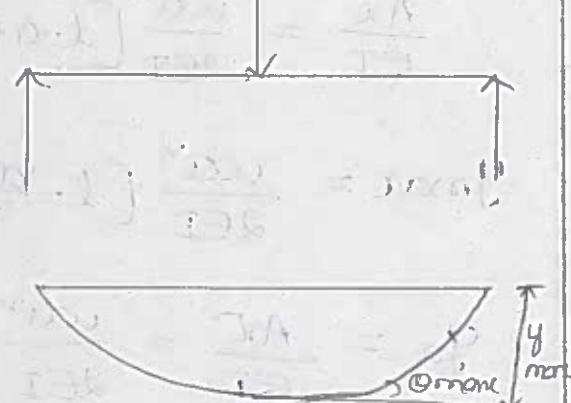
$$\Theta_{\max} = \frac{\omega l^2}{(G E I)}$$

$$\text{and } y_{\max} = \frac{A \bar{x}}{E I}$$

$$= \frac{\omega l^3}{(G E I)} \times \frac{2}{3} \left(\frac{l}{2}\right)$$

$$y_{\max} = \frac{\omega l^3}{48 E I}$$

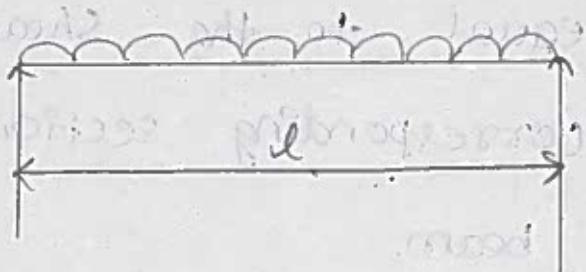
$$y_{\max} = \frac{\omega l^3}{48 E I}$$



case(5): Simply Supported Beam with Uniformly Distributed Load :-

$$\Theta_{max} = \frac{A}{EI}$$

A = Area of shaded part

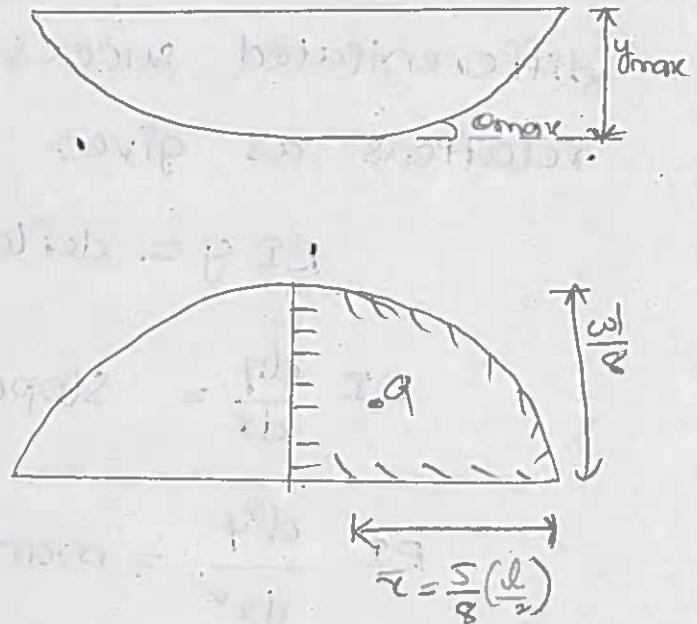


$$= \frac{2}{3} \cdot \frac{l}{2} \cdot \frac{wl}{3}$$

$$= \frac{wl^3}{24}$$

$$\Theta_{max} = \frac{wl^2}{24EI}$$

$$\text{and } y_{max} = \frac{Ax}{EI}$$



$$= \frac{wl^3}{24EI} \times \frac{5}{8} \left(\frac{l}{2}\right)$$

$$y_{max} = \frac{5wl^3}{384EI} \quad [E: x = \frac{5}{8} \left(\frac{l}{2}\right)]$$

Conjugate Beam Method : conjugate beam is an imaginary beam of length equal to that of the original beam but for which the load diagram is the $\frac{M}{EI}$ diagram

i.e. the load at any point on the conjugate beam is equal to B.M at that point divided by EI)

The slopes and deflection at any section of a beam by conjugate

beam method is given by the slope at any section of the given beam is equal to the shear force at the corresponding section of the conjugate beam.

When the deflection equation is differentiated successively, we get the relations as given below

$$EIy = \text{deflection}$$

$$EI \frac{dy}{dx} = \text{slope}$$

$$EI \frac{d^2y}{dx^2} = \text{moment} = M$$

$$EI \frac{d^3y}{dx^3} = \text{shear} = S = -\frac{dm}{dx}$$

$$EI \frac{d^4y}{dx^4} = \text{load} = \frac{ds}{dx} = \frac{d^2M}{dx^2}$$

In this method, the beam (known as conjugate beam), is loaded with not the actual loads, but with the elastic weight $\frac{M}{EI}$ corresponding to the actual load.

Conjugate Beam Theorem I :- "The slope at any section of a loaded beam relative to the original axis of beam at the corresponding section.

(21) we know that

$$\text{load} = w = \frac{M}{EI}$$

$$\text{Shear} = S_x = \int_0^x w \cdot dx = \int_0^x \frac{M}{EI} dx$$

$$\text{But } \int_0^x \frac{M}{EI} dx = \int_0^x \frac{dy}{dx} dx = \text{slope}$$

conjugate Beam Theorem II:

"The deflection at any given section of a loaded beam, relative to the original position, is equal to the bending moment at the corresponding section of the conjugate beam."

$$\text{we know that, Shear } S_x = \int_0^x \frac{M}{EI} dx$$

$$\therefore \text{Bending moment } M_x = \int_0^x S_x dx = \int_0^x \int_0^x \frac{M}{EI} dx$$

$$\text{But } \int_0^x \int_0^x \frac{M}{EI} dx = \int_0^x \int_0^x \frac{dy}{dx} dx + \int_0^x \frac{dy}{dx} dx = y = \text{deflection.}$$

The following points are worth nothing for the conjugate beam method.

- i) This method can be directly used only for simply supported beams.
- ii) In this method for cantilevers and fixed beams, artificial constraints bend to be

applied to the conjugate beam so that it is supported in a manner consistent with the constraints of the real beam.

Relation between Actual Beam And Conjugate Beam

Real Actual beam

conjugate beam.

1. simply supported or roller, simply supported end supported end (deflection $\neq 0$ but S.F exists) B.M = 0 but S.F exists.
= 0 but slope exists) Fixed end (S.F and B.M exist)
2. free end (slope and deflection ex.st) fixed end (S.F and B.M exist)
3. fixed end (slope and deflection are zero) free end (S.F and B.M are zero)
4. slope at any section S.F at the corresponding section.
5. deflection at any section B.M at the corresponding section. the
6. Given system of loading. The loading diagram F.S M/EI diagram.
7. B.M diagram positive (sagging) M/EI load diagram is positive (i.e. loading is down ward).
8. B.M diagram negative (hogging) M/EI load diagram is negative (i.e. loading is upward).

Deflection And Slope Of A S.S.B with A point Load At the centre

Fig shows a S.S.B AB of length L carrying a point load w at the centre C. The B.M at A and B is zero and at the centre B.M will be $wL/4$. The B.M varies according to straight line law.

The load on the

conjugate beam will

be obtained by

dividing the B.M

at that point

by EI

Let R_A = Reaction

at A for conjugate beam.

R_B = Reaction at B

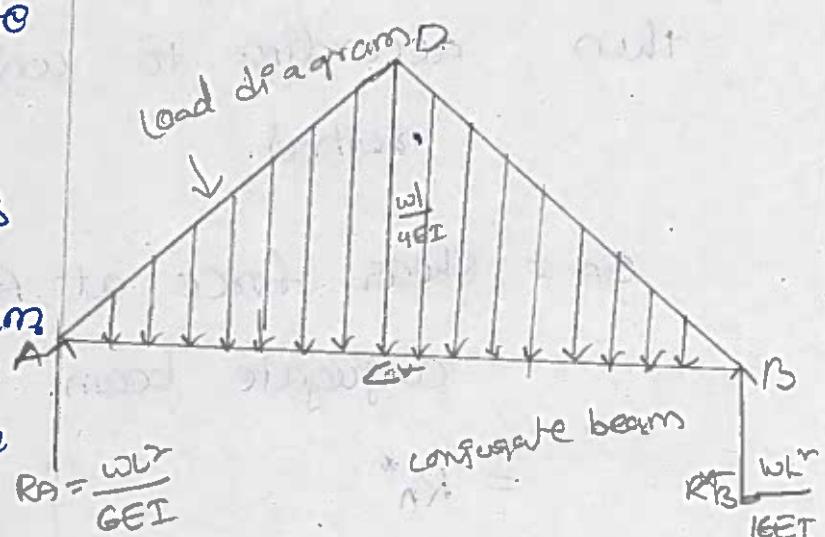
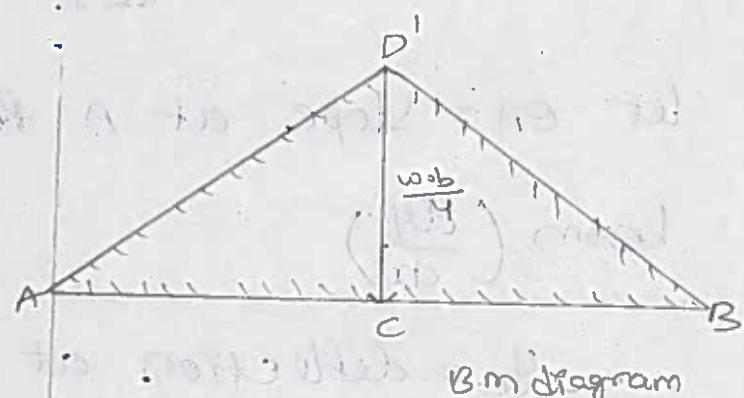
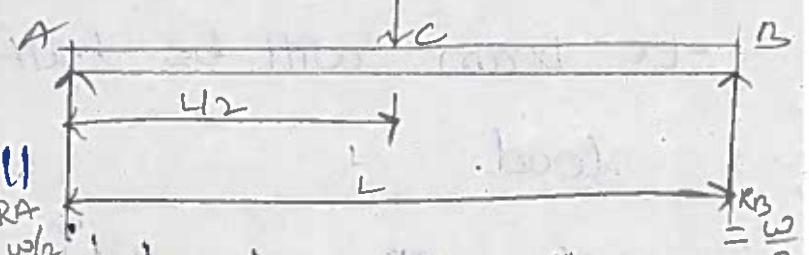
for conjugate beam

The load on the

conjugate beam

from the fig can

be written as



= Area of the load diagram

$$= \frac{1}{2} \times AB \times CD$$

$$= \frac{1}{2} \times L \times \frac{wL}{4EI}$$

$$= \frac{wL^3}{8EI}$$

Reaction at each support for the conjugate beam will be half of the total load.

$$\therefore R_A^* = R_B^* = \frac{1}{2} \times \frac{wL^3}{8EI} = \frac{wL^3}{16EI}$$

let Θ_A = slope at A for the given beam ($\frac{dy}{dx}$)

y_C = deflection at 'C'

then according to conjugate beam method.

Θ_A = shear force at A for the conjugate beam.

$$= R_A^*$$

$$\Theta_A = \frac{wL^3}{16EI}$$

and y_c = B.M at 'c' for the conjugate beam.

$$= R^* A \times \frac{L}{2} - \text{load corresponding to } AC^* O^* X$$

distance of C.G. of AC_0^{*} from 'C'

$$= \frac{\omega L^3}{16EI} \times \frac{L}{2} - \left[\frac{1}{2} \times \frac{L}{2} \times \frac{\omega L}{4EI} \right] \times \left[\frac{1}{3} \times \frac{L}{2} \right]$$

$$= \frac{\omega L^3}{32EI} - \frac{\omega L^3}{96EI}$$

$y_C = \frac{\omega L^3}{48EI}$

Deflection And Slope of A Cantilever with A point Load At the free End :-

fig shows a cantilever

AB of length 'l' and

carrying a point

load w at the

free end B. The B.M.

is zero at the

free end B and $B.M_C$

at A is equal to

$$w \cdot L$$

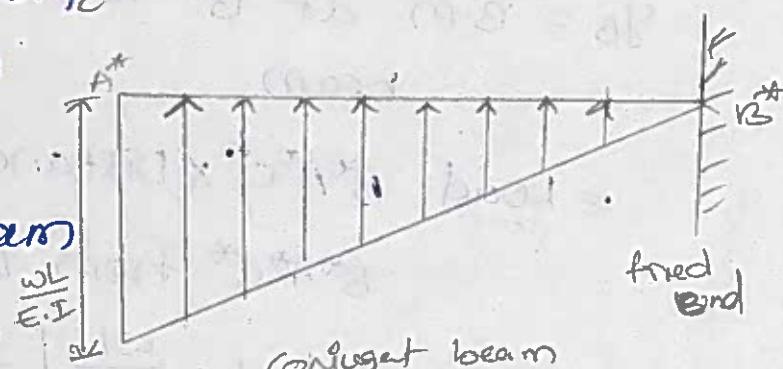
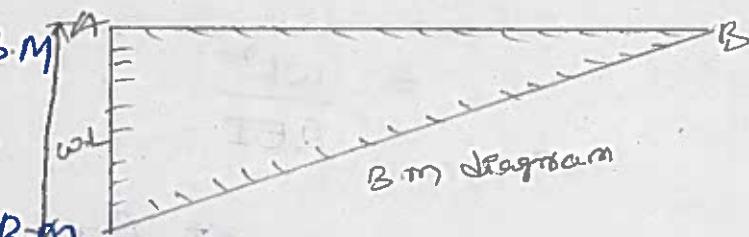
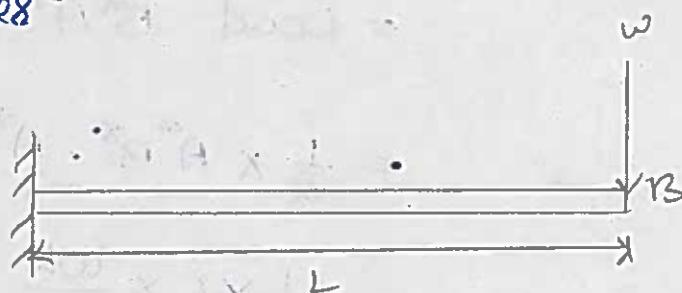
The conjugate beam

can be drawn by

dividing the B.M

at any section by

$$EI$$



the loading on the conjugate beam will be negative (i.e upwards) as B.M for cantilever is negative the loading on conjugate beam is shown in fig.

let Θ_B = slope at B i.e $(\frac{dy}{dx})$ at B.

y_B = deflection at B.

then according to the conjugate beam method.

Θ_B = S.F at B^* for the conjugate beam.

= Load $B^*A^*C^*$

$$= \frac{1}{2} \times A^*B^* \cdot A^*C^*$$

$$= \frac{1}{2} \times L \times \frac{\omega L}{EI}$$

$$= \frac{\omega L^2}{2EI}$$

y_B = B.M at B^* for the conjugate beam

= Load $B^*A^*C^* \times$ Distance of C.G of $B^*A^*C^*$ from B^*

$$= \frac{1}{2} \cdot L \cdot \frac{\omega L}{EI} \left[\frac{2}{3} : L \right]$$

$$y_B = \frac{\omega L^3}{3EI}$$